

Quantum Fisher information of fermionic cavity modes in an accelerated motion

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We investigate the effect of the inertial and non-inertial segments of relativistic motion on the quantum Fisher information of $(1+1)$ Dirac field modes confined to cavities. For the purpose, we consider the situation that Rob's cavity, initially inertial, accelerates uniformly with respect to its proper time and then again becomes inertial while Alice's cavity remains inertial. The acceleration is assumed to be very small and its effects were analyzed in a perturbative regime. For analysis, we consider θ parameterized two-qubit pure entangled state and a Werner state. In contrast to the degradation of entanglement due to the relativistic motion between the cavities, the quantum Fisher information of the pure composite system \mathcal{F}_θ with respect to parameter θ is found to be invariant under the same conditions. However, in the case of the Werner state, the quantum Fisher information displays periodic degradation, due to the inertial and non-inertial segments of motion. Further, we investigate how this evolution process affects the quantum Fisher information distribution over the subsystems of Alice's and Rob's cavities. We find that the quantum Fisher information over the Rob's cavity shows the periodic degradation behavior depending upon the parameter θ as well as the uniform acceleration for both the two qubit pure state and Werner state. The quantum Fisher information over Alice's cavity remains invariant throughout the motion of Rob's cavity for the two qubit pure state whereas for Werner state it is affected by the mixing parameter of the Werner state.

I. INTRODUCTION

In addition to well established measures like quantum entanglement and quantum discord in quantum information, quantum Fisher information (QFI) [1–3] has recently gained considerable attention as a promising candidate to understand the information contents of quantum states. In fact it is well known that QFI has been successfully applied to quantum statistical inference and estimation theory [4, 5]. When information is encrypted in a set of states, more often in terms of appropriate parameters, extraction of the encoded information requires the distinction between different states. This requirement can be accomplished through measurements that yield a set of observations characterized by those parameters. Therefore the problem of distinguishing the states is transformed into a problem of parameter estimation. In such situations, the classical Fisher information plays a pivotal role as it provides the lower bound on the variance of the parameter estimation in terms of the well known Cramér-Rao bound. For asymptotically large sets of observations, the maximum likelihood estimation (MLE) approach provides unbiased parameter estimation that can attain this bound [6]. However, the situation becomes more involved when we delve into the realm of information extraction from quantum states. This is because the observations are the results of quantum measurements. In this scenario, quantum information content is much more complex and intrinsically contains non-trivial characteristics that can not be associated with their classical counterpart [1, 7]. In order to deal with the quantum information based applications, the quantum Fisher information was coined [2, 3, 7] as a natural extension to the classical Fisher information.

Furthermore, successful efforts to combine relativity and quantum information theory have opened new avenues to explain quantum behavior at a macroscopic scale [8–12]. A focus of this area of research is the dynamics of information contents under the Unruh-Hawking effect [13–15]. The dynamics of the other important quantum information resources like entanglement [16–20], quantum discord [21–23], fidelity of teleportation [24] have already been studied. For instance the correlations induced by maximally entangled bosonic or fermionic bipartite states were found to be degraded due to the accelerated motion of one party with respect to the other inertial party [17, 19].

In fact, the study of Unruh-Hawking effect on the quantum Fisher information is a crucial step to use QFI as a reliable resource for quantum metrology in a relativistic regime. Aspachs et al. described the optimal detection process of Unruh-Hawking effect itself and showed that the Fock states can achieve maximal QFI with a scalar field in a two-dimensional Minkowski spacetime [25]. Yao et al. [26] investigated the performance of QFI for both scalar

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and Dirac fields parameterized by weight and phase parameters in non-inertial frames. Ahmadi et al. [27] proposed a framework to evaluate the QFI of the acceleration parameter for a relativistic scalar field confined to a cavity using Bures Statistical distance [28].

In this work, we investigate the behavior of quantum Fisher information for a parameterized Dirac field restricted in the cavities. Following the Dirac field analysis proposed in [19], the modes of massless Dirac field are confined to the two cavities with Dirichlet's boundary conditions, where one of the cavities remains inertial while the other undergoes the segments of inertial and non-inertial motion with uniform acceleration. We restrict the uniform acceleration to be very small ($a \ll 1$) and use perturbation theory to observe the effects of the accelerated motion on the variation of the QFI for the confined modes of the Dirac field with respect to the weight parameter θ .

The rest of the paper is organized as follows. In Sec. II, we briefly review the properties of QFI and discuss the recent developments regarding its analytic evaluation. The perturbed Bogoliubov coefficients and Fock space quantization for vacuum and one charged particle fermionic states in the cavities are described in Sec. III. The QFI for two mode fermionic Fock state shared between Alice's and Rob's cavities and distribution of QFI over each cavity is discussed in Sec. IV. In Sec. V, we extend our investigation to Werner state. In Sec. VI, we conclude and discuss our results.

II. QUANTUM FISHER INFORMATION

Consider a quantum state ρ_λ having a one parameter family of N-dimensional quantum states, with parameter λ . A set of quantum measurements $\{E(\xi)\}$, which is POVM, should be performed on ρ_λ to extract the information about unknown parameter λ . This process resembles the classical case where the parameter estimation is performed from the observed data obtained from classical experiments and the efficiency of the estimate(s) is measured in terms of the classical Fisher information [3, 6, 7]

$$F_\lambda = \int d\xi p(\xi|\lambda) [\partial_\lambda \ln p(\xi|\lambda)]^2 = \int d\xi \frac{(\partial_\lambda p(\xi|\lambda))^2}{p(\xi|\lambda)}, \quad (1)$$

where $p(\xi|\lambda)$ denotes the conditional probability of observing the result ξ provided the value of the parameter is λ . However, the quantum Fisher information differs significantly in the operational sense due the fact that the probability $p(\xi|\lambda)$ results from measurement by the quantum operator $E(\xi)$ rather than a mere classical experiment. According to Born's rule, $p(\xi|\lambda) = \text{Tr}[E(\xi)\rho_\lambda]$ represents the conditional probability of obtaining the outcome ξ when the given value of the parameter is λ . In the quantum world, the parameter estimation problem can be thought of as the problem of searching for the set of measurements $\{E(\xi)\}$ that yields the optimal value of (1). Therefore, the quantum Fisher information can be defined as [1, 2]

$$\mathcal{F}_\lambda = \max_{E(\xi)} F_\lambda. \quad (2)$$

Further, using the notion of symmetric logarithmic derivative (SLD) introduced in [3]

$$\partial_\lambda \rho_\lambda = \frac{1}{2} (\rho_\lambda L_\lambda + L_\lambda \rho_\lambda), \quad (3)$$

we obtain an alternate expression for the QFI as

$$\mathcal{F}_\lambda = \text{Tr}(\rho_\lambda L_\lambda^2) = \text{Tr}(\partial_\lambda \rho_\lambda L_\lambda), \quad (4)$$

where the operator L_λ is the solution of the Lyapunov matrix equation [29] given by Eq. (3). It is worth noticing that expressions of QFI in (2) and (4) are equivalent in the sense that the operator L_λ admits the spectral decomposition in the same basis as used for the spectral decomposition of ρ_λ . Therefore, based on the spectral decomposition $\rho_\lambda = \sum_{i=1}^N p_i |\psi_i\rangle\langle\psi_i|$, the operator L_λ is written as [29]

$$L_\lambda = 2 \sum_{m,n}^N \frac{|\langle \psi_m | \partial_\lambda \rho_\lambda | \psi_n \rangle|}{p_m + p_n} |\psi_m \times \psi_n|, \quad (5)$$

where the eigenvalues $p_i \geq 0$ and $\sum_{i=1}^N p_i = 1$. In turn, the set of eigenvectors of L_λ provides the optimal POVM to obtain the QFI. Making use of (4) and (5), the QFI can be rewritten as [29]

$$\mathcal{F}_\lambda = 2 \sum_{m,n}^N \frac{|\langle \psi_m | \partial_\lambda \rho_\lambda | \psi_n \rangle|^2}{p_m + p_n}. \quad (6)$$

Using the completeness relation

$$\sum_{i=M+1}^N |\psi_i \times \psi_i| = \mathbb{I} - \sum_{i=1}^M |\psi_i \times \psi_i|, \quad (7)$$

where M denotes the number of eigenvectors corresponding to non-zero eigenvalues and represents the dimension of the support of ρ_λ , Zhang et al. provided a relatively simple expression to evaluate the QFI for general states as [30]

$$\mathcal{F}_\lambda = \sum_{i=1}^M \frac{(\partial_\lambda p_i)^2}{p_i} + \sum_{i=1}^M 4p_i \mathcal{F}_{\lambda,i} - \sum_{i \neq j}^M \frac{8p_i p_j |\langle \psi_i | \partial_\lambda \psi_j \rangle|^2}{p_i + p_j}, \quad (8)$$

where $\mathcal{F}_{\lambda,i}$ is the QFI of the pure state $|\psi_i\rangle$ given by

$$\mathcal{F}_{\lambda,i} = \langle \partial_\lambda \psi_i | \partial_\lambda \psi_i \rangle - |\langle \psi_i | \partial_\lambda \psi_i \rangle|^2. \quad (9)$$

It is noteworthy that the QFI of a low rank density matrix is now only determined by its support spanned by the eigenvectors, $|\psi_i\rangle$; $i = 1, \dots, M$, corresponds to nonzero eigenvalues. Furthermore, from (8), three contributions to QFI can be identified: the first term on the right hand side of (8) represents the classical case where the set of nonzero eigenvalues behaves like a classical probability distribution; the second contribution denotes the weighted average of QFI due to all the pure-states; the last term captures the QFI from the mixture of pure states and therefore causes the decrease in the total QFI. Therefore, the last two terms can be interpreted as the quantum part of the Fisher information while the first term represents its classical constituent. In this paper, we will exploit Eq. (8) due to its relative simplicity and clear physical interpretation to find the QFI for the modes of the Dirac field confined to the inertial and non-inertial cavities.

III. BOGOLIUBOV TRANSFORMATION FOR INERTIAL AND NON-INERTIAL SEGMENTS

In order to find the unitary transformation of cavity's transitions between the inertial and non-inertial segments of motion, we setup two cavities with the observers referred to as Alice and Rob, respectively. Both the cavities are inertial and completely overlap at $t = 0$. Here we assume that the cavity walls are placed at $x = a$ and $x = b$ where $0 < a < b$. Rob's cavity then moves with uniform acceleration to the right along the time-like killing vector ∂_η for duration $\eta = 0$ to $\eta = \eta_1$ in the Rindler co-ordinates. Duration of the acceleration, $\frac{2}{a+b}$, with respect to proper time measured at the center of the cavity is thus $\tau_1 = \frac{a+b}{2}\eta_1$. Finally, Rob's cavity again becomes inertial with respect to its rest frame. The Alice's cavity remains inertial throughout this trip of Rob's cavity. Therefore, three segments of Rob's trajectories can be identified as Regions I, II and III.

This grafting process of cavity motion is explained here to make this paper sufficiently self contained. Earlier the same process is introduced and exploited by [19, 31] to study the dynamics of the entanglement for bosonic and fermionic cavities.

The Dirac field representation in three regions is

$$\text{I: } \psi = \sum_{n \geq 0} a_n \psi_n + \sum_{n < 0} b_n^\dagger \psi_n, \quad (10a)$$

$$\text{II: } \psi = \sum_{n \geq 0} \hat{a}_n \hat{\psi}_n + \sum_{n < 0} \hat{b}_n^\dagger \hat{\psi}_n, \quad (10b)$$

$$\text{III: } \psi = \sum_{n \geq 0} \tilde{a}_n \tilde{\psi}_n + \sum_{n < 0} \tilde{b}_n^\dagger \tilde{\psi}_n, \quad (10c)$$

with the respective non vanishing anticommutators

$$\text{I: } \{a_m, a_n^\dagger\} = \{b_m, b_n^\dagger\} = \delta_{mn}, \quad (11a)$$

$$\text{II: } \{\hat{a}_m, \hat{a}_n^\dagger\} = \{\hat{b}_m, \hat{b}_n^\dagger\} = \delta_{mn}, \quad (11b)$$

$$\text{III: } \{\tilde{a}_m, \tilde{a}_n^\dagger\} = \{\tilde{b}_m, \tilde{b}_n^\dagger\} = \delta_{mn}. \quad (11c)$$

Using Bogoliubov transformation, the Dirac field modes between Region I and II are related as [19]

$$\hat{\psi}_m = \sum_n A_{mn} \psi_n, \quad (12)$$

where ψ_n and $\hat{\psi}_m$ are the Dirac field modes in regions I and II respectively. For the small acceleration case, Friis et al. [19] derived these coefficients A_{mn} in the perturbative regime by introducing the dimensionless parameter $h = \frac{2L}{a+b}$, satisfying $0 < h < 2$. These coefficients preserve the unitarity of the transformation to the order $O(h^2)$ for $s > 0, s \rightarrow 0_+$ and are given in terms of Maclaurin's series expansion as [19, 31]

$$A_{mn} = A_{mn}^{(0)} + A_{mn}^{(1)} + A_{mn}^{(2)} + O(h^3), \quad (13)$$

where the superscripts indicate the power of parameter h . During the non-inertial trajectory, modes $\hat{\psi}_m$ in Rob's cavity remain independent and do not interact. Hence, these modes can only develop some phases during the non-inertial duration $0 \leq \eta \leq \eta_1$. This change in the modes can be balanced by introducing a diagonal matrix $G(\eta_1)$ whose diagonal entries are [19]

$$G_{nn}(\eta_1) = \exp(i\Omega_n\eta_1). \quad (14)$$

For $\eta \geq \eta_1$, the transformation from region II to region III can be obtained by simply using the inverse transformation $A^\dagger = A^{-1}$. The evolution of the Dirac field mode from region I to region III in Rob's cavity can then be expressed by the Bogoliubov transformation matrix

$$\mathcal{A} = A^\dagger G(\eta_1) A. \quad (15)$$

Thus the Bogoliubov transformation for the Dirac field modes between regions I and III reads

$$\tilde{\psi}_m = \sum_n \mathcal{A}_{mn} \psi_n. \quad (16)$$

It can be noticed that \mathcal{A} , being the composition of unitary matrices, is also a unitary matrix to the order h^2 .

Similarly, the Bogoliubov transformation for the Dirac field mode operators can also be expressed as [19, 31]

$$k > 0 : \quad a_k = \sum_{l \geq 0} \tilde{a}_l \mathcal{A}_{lk} + \sum_{l < 0} \tilde{b}_l^\dagger \mathcal{A}_{lk}, \quad (17a)$$

$$k < 0 : \quad b_k = \sum_{l \geq 0} \tilde{a}_l^\dagger \mathcal{A}_{lk} + \sum_{l < 0} \tilde{b}_l \mathcal{A}_{lk}. \quad (17b)$$

The relation between the Fock vacua in regions I and III denoted by $|0\rangle$ and $|\tilde{0}\rangle$ is [11, 19]

$$|0\rangle = N e^W |\tilde{0}\rangle, \quad (18)$$

where

$$W = \sum_{p \geq 0, q < 0} V_{pq} \tilde{a}_p^\dagger \tilde{b}_q^\dagger. \quad (19)$$

The coefficient matrix V and the normalization constant N are the unknowns to be evaluated. Using (10a),(10c), and (16), the coefficient matrix is given by

$$V = V^{(0)} + V^{(1)} + O(h^2) = V^{(1)} + O(h^2), \quad (20)$$

with

$$V_{pq}^{(1)} = \mathcal{A}_{pq}^{(1)*} G_q = -\mathcal{A}_{qp}^{(1)} G_p^* \quad (21)$$

The relation between Fock vacua in regions I and III given by (18) yields [19, 31]

$$\begin{aligned} |0\rangle = & \left(1 - \frac{1}{2} \sum_{p \geq 0, q < 0} |V_{pq}|^2\right) |\tilde{0}\rangle + \sum_{p,q} V_{pq} |\tilde{1}_p\rangle^+ |\tilde{1}_q\rangle^- \\ & - \frac{1}{2} \sum_{p,q} \sum_{i,j} V_{pq} V_{ij} \phi_{p,i} \phi_{q,j} |\tilde{1}_p\rangle^+ |\tilde{1}_i\rangle^+ |\tilde{1}_q\rangle^- |\tilde{1}_j\rangle^- + O(h^3). \end{aligned} \quad (22)$$

where $|\tilde{1}_p\rangle^+ := \tilde{a}_p^\dagger |\tilde{0}\rangle^+$ and $|\tilde{1}_q\rangle^- := \tilde{b}_q^\dagger |\tilde{0}\rangle^-$ represent the single-particle Fock states for modes $p \geq 0$ and $q < 0$, respectively. The sign \pm in the superscript denotes the sign of the charge. Further, the term $\phi_{pi} := 1 - \delta_{p,i}$ is introduced to incorporate the Pauli-exclusion principle for the single particle states with same charge sign. Also the ordering of the single-particle kets corresponds to the ordering of the fermionic creation operators rather than the fermionic modes [31]. Similarly, the charged single particle states in region III are [19, 31]

$$\begin{aligned}
k > 0: \quad |1_k\rangle^+ &= - \sum_{p,q} V_{pq} \mathcal{A}_{qk}^* |\tilde{1}_p\rangle^+ + \sum_{m \geq 0} \mathcal{A}_{mk}^* \left\{ \left(1 - \frac{1}{2} \sum_{p,q} |V_{pq}|^2\right) |\tilde{1}_m\rangle^+ + \sum_{p,q} V_{pq} \phi_{pm} |\tilde{1}_m\rangle^+ |\tilde{1}_p\rangle^+ |\tilde{1}_q\rangle^- \right. \\
&\quad \left. - \frac{1}{2} \sum_{p,q;i,j} V_{pq} V_{ij} \phi_{pi} \phi_{pm} \phi_{mi} \phi_{qj} |\tilde{1}_m\rangle^+ |\tilde{1}_p\rangle^+ |\tilde{1}_i\rangle^+ |\tilde{1}_q\rangle^- |\tilde{1}_j\rangle^- \right\} + O(h^3), \tag{23a}
\end{aligned}$$

$$\begin{aligned}
k < 0: \quad |1_k\rangle^- &= \sum_{p,q} V_{pq} \mathcal{A}_{pk} |\tilde{1}_q\rangle^- + \sum_{m < 0} \mathcal{A}_{mk} \left\{ \left(1 - \frac{1}{2} \sum_{p,q} |V_{pq}|^2\right) |\tilde{1}_m\rangle^- + \sum_{p,q} V_{pq} \phi_{qm} |\tilde{1}_p\rangle^+ |\tilde{1}_q\rangle^- |\tilde{1}_m\rangle^- \right. \\
&\quad \left. - \frac{1}{2} \sum_{p,q;i,j} V_{pq} V_{ij} \phi_{pi} \phi_{qm} \phi_{qj} \phi_{mj} |\tilde{1}_p\rangle^+ |\tilde{1}_i\rangle^+ |\tilde{1}_q\rangle^- |\tilde{1}_j\rangle^- |\tilde{1}_m\rangle^- \right\} + O(h^3), \tag{23b}
\end{aligned}$$

where the one particle states $|1_k\rangle^\pm$ in region I are

$$k \geq 0: \quad |1_k\rangle^+ = a_k^\dagger |0\rangle, \tag{24a}$$

$$k < 0: \quad |1_k\rangle^- = b_k^\dagger |0\rangle. \tag{24b}$$

IV. QUANTUM FISHER INFORMATION FOR TWO-MODE STATES

Here we investigate the behavior of the QFI for a complete trip from region I to region III in the perturbative regime to the order h^2 .

A. Evaluation of Perturbed Eigenvalues and Eigenvectors

We consider a bipartite two qubit pure state parameterized by weight parameter θ in region I. The state consists of two Dirac field modes where one of the modes is confined to Alice's cavity and the other is confined to Rob's cavity. The initial parameterized state is

$$|\psi_{init}^\pm\rangle_\nu = \cos \theta |0\rangle_A |0\rangle_R \pm \sin \theta |1_m\rangle_A^\mu |1_k\rangle_R^\nu, \tag{25}$$

where the subscripts A and R refer to the cavity of Alice and Bob, respectively. The superscripts μ and ν indicate whether the mode has positive or negative frequency, so that $\mu(\nu) = +$ for $m(k) \geq 0$ and $\mu(\nu) = -$ for $m(k) \leq 0$. Following the procedure given in [19, 31], the initial state (25) is represented by the two-particle basis of the two mode Hilbert space with one excitation for each of the modes m and k in Alice's and Rob's cavities, respectively. The corresponding density matrix becomes

$$\begin{aligned}
\rho_\nu^\pm &= \left[\cos^2 \theta |0\rangle_A \langle 0| \otimes |0\rangle_R \langle 0| + \sin^2 \theta |1_m\rangle_A^{\mu\mu} \langle 1_m| \right. \\
&\quad \left. \otimes |1_k\rangle_R^{\nu\nu} \langle 1_k| + (\sin \theta \cos \theta |0\rangle_A^\mu \langle 1_m| \otimes |0\rangle_R^\nu \langle 1_k| \right. \\
&\quad \left. + h.c.) \right]. \tag{26}
\end{aligned}$$

It should be noted that a partial trace is taken over all of Rob's modes except the reference mode k . Using the inside out partial tracing approach [19], the reduced density matrix in region III is

$$\begin{aligned}
\text{Tr}_{\neg k} \rho_\nu^\pm &\equiv \rho_{\nu,k}^\pm = \cos^2 \theta |0\rangle_A \langle 0| \otimes \left\{ (1 - f_k^{-\nu} h^2) |\tilde{0}\rangle_{III} \langle \tilde{0}| + f_k^{-\nu} h^2 |\tilde{1}_k\rangle_{III} \langle \tilde{1}_k| \right\} + \sin \theta \cos \theta \left\{ \pm \left(G_k + \mathcal{A}_{kk}^{(2)} h^2 \right) \right. \\
&\quad \left. \times |0\rangle_A^\mu \langle 1_m| \otimes |\tilde{0}\rangle_{III}^\nu \langle \tilde{1}_k| \pm h.c. \right\} + \sin^2 \theta |1_m\rangle_A^{\mu\mu} \langle 1_m| \otimes \left\{ (1 - f_k^\nu h^2) |\tilde{1}_k\rangle_{III}^{\nu\nu} \langle \tilde{1}_k| + f_k^\nu h^2 |\tilde{0}\rangle_{III} \langle \tilde{0}| \right\}. \tag{27}
\end{aligned}$$

where f_k^ν and $f_k^{-\nu}$ are defined as

$$\nu > 0: \quad f_k^\nu = \sum_{p \geq 0} |\mathcal{A}_{pk}^{(1)}|^2, \tag{28a}$$

$$\nu < 0: \quad f_k^\nu = \sum_{q < 0} |\mathcal{A}_{qk}^{(1)}|^2. \tag{28b}$$

The density matrix can be re-written as

$$\rho_{\nu,k}^{\pm} = \rho_{\nu,k}^{\pm(0)} + \rho_{\nu,k}^{\pm(2)} h^2, \quad (29)$$

where $\rho_{\nu,k}^{\pm(0)}$ and $\rho_{\nu,k}^{\pm(2)}$ denote the unperturbed and perturbed matrix components, respectively. In order to evaluate the QFI given by (8) for the density matrix in (27), we compute the non-zero eigenvalues and corresponding eigenvectors. With the help of perturbation theory [32, 33], the eigenvalues of the perturbed density matrix thus become

$$\text{EigenVal}(\rho_{\nu,k}^{\pm}) = \{p_i\} = \left\{ 1 - (\cos^2 \theta f_k^{-\nu} + \sin^2 \theta f_k^{\nu}) h^2, \right. \\ \left. \cos^2 \theta f_k^{-\nu} h^2, \sin^2 \theta f_k^{\nu} h^2, 0 \right\}. \quad (30)$$

Since the trace of the perturbed density matrix is $\text{tr}(\rho_{\nu,k}^{\pm}) = 1$ and all the eigenvalues are non-negative, $\rho_{\nu,k}^{\pm}$ satisfies the density matrix representation. Again, using perturbation theory, the normalized eigenvectors corresponding to the non-zero eigenvalues of $\rho_{\nu,k}^{\pm}$ are

$$|\Phi_1\rangle = \frac{1}{\sqrt{N}} \left\{ G_k^{(\nu_*)} (\cos \theta - \alpha \sin \theta), 0, 0, (\sin \theta + \alpha \cos \theta) \right\}, \quad (31a)$$

$$|\Phi_2\rangle = \{0, 1, 0, 0\}, \quad (31b)$$

$$|\Phi_3\rangle = \{0, 0, 1, 0\}, \quad (31c)$$

where α and the normalization constant N are found to be

$$\alpha := \sin \theta \cos \theta \left(\frac{f_k^{-\nu} - f_k^{\nu}}{2} + i \text{Im} \left(G_k \bar{A}_{kk}^{(2)} \right) \right) h^2, \quad (32)$$

$$N = 1 + |\alpha|^2. \quad (33)$$

B. Quantum Fisher Information with respect to the weight parameter

Next, we use (8) to compute the QFI of the Dirac field evolved from region I to region III. Prior to the evolution, the QFI for the Dirac field of the initial bipartite system (26) with respect to weight parameter θ is $\mathcal{F}_{\theta} = 4$. After the evolution of the system to the region III, we now compute quantum and classical contribution separately. By using Eq. (9), the contribution to the QFI due to $|\Phi_1\rangle$ is

$$\mathcal{F}_{\theta,1} = \langle \Phi'_1 | \Phi'_1 \rangle - |\langle \Phi_1 | \Phi'_1 \rangle|^2 \\ = 1 + \cos 2\theta (f_k^{-\nu} - f_k^{\nu}) h^2 + O(h^3), \quad (34)$$

where we have used (32) and (33) to get

$$\frac{\text{Re}(\alpha')}{N} = \cos 2\theta \frac{f_k^{-\nu} - f_k^{\nu}}{2} h^2 + O(h^3). \quad (35)$$

Here, we see that the other two states namely $|\Phi_2\rangle$ and $|\Phi_3\rangle$ are independent of the parameter θ . Therefore, the individual contributions $\mathcal{F}_{\theta,2}$ and $\mathcal{F}_{\theta,3}$ disappear. Following the same argument, the mixed terms $\langle \Phi_m | \Phi'_n \rangle$ disappear as well $\forall m \neq n$. Consequently, the net quantum contribution, F_{θ} , to the total QFI (8) is computed as

$$F_{\theta} = 4p_1 \mathcal{F}_{\theta,1} \\ = 4 (1 - \cos^2 \theta f_k^{-\nu} h^2 - \sin^2 \theta f_k^{\nu} h^2 + O(h^3)) \\ \times (1 + \cos 2\theta (f_k^{-\nu} - f_k^{\nu}) h^2 + O(h^3)) \\ = 4 (1 - (\sin^2 \theta f_k^{-\nu} - \cos^2 \theta f_k^{\nu}) h^2) + O(h^3). \quad (36)$$

Working perturbatively in the similar fashion and using the following relations

$$\frac{(p'_1)^2}{p_1} = O(h^3), \quad (37a)$$

$$\frac{(p'_2)^2}{p_2} = 4 \sin^2 \theta f_k^{-\nu} h^2 + O(h^3), \quad (37b)$$

$$\frac{(p'_3)^2}{p_3} = 4 \cos^2 \theta (f_k^{\nu}) h^2 + O(h^3), \quad (37c)$$

the classical contribution, F_c , to the QFI (8) is given by

$$\begin{aligned} F_c &= \sum_{i=1}^3 \frac{(p'_i)^2}{p_i} \\ &= 4 (\sin^2 \theta f_k^{-\nu} + 4 \cos^2 \theta f_k^{\nu}) h^2. \end{aligned} \quad (38)$$

Finally, using (36) and (38), the QFI (8) of the Dirac field modes confined to Alice's and Rob's cavities reads

$$\mathcal{F}_\theta = F_\theta + F_c = 4, \quad (39)$$

where Rob's field mode has evolved as discussed before. It is worth noticing that to the order h^2 , the QFI of the bipartite cavities system remains invariant even after the evolution of field mode in Rob's cavity. In contrast to QFI, the entanglement of the maximally entangled bipartite system ($\theta = \frac{\pi}{4}$ in Eq. (27)) was found to have periodic degradation under the same evolution process [19]. The periodic degradation of the entanglement can be avoided by fine tuning the duration of inertial and non-inertial segments of Rob's cavity motion. However, no such fine tuning is needed in case of QFI with respect to the weight parameter θ .

Furthermore the invariance of \mathcal{F}_θ through the inertial and non-inertial segments of motion is highly nontrivial as for an arbitrary single-qubit state, Zhong et al. [34] have indicated that \mathcal{F}_θ remains unchanged only through phase-damping channel. The QFI with respect to parameter θ for scalar and Dirac field modes, which are not confined to the cavities, under Unruh-Hawking effect [26], yields the same result as that in our case.

C. Quantum Fisher Information distribution over subsystems

Lu et al. [35] provided an elegant hierarchical analysis for the QFI of the composite system with respect to its constituent subsystems for a variety of measurement settings. In a similar fashion, we study the effect of accelerated motion of Rob's cavity on the distribution of the QFI, \mathcal{F}_θ , over constituent subsystems. For the inertial fermionic cavity of Alice, it is straightforward to show that $\mathcal{F}_\theta^A = 4$. Now let us consider the case for the fermionic mode in

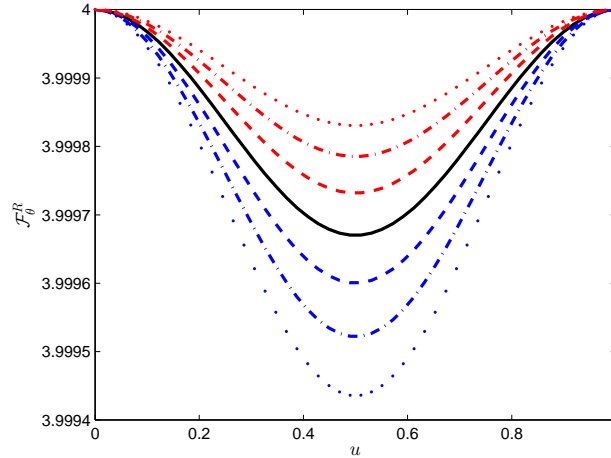


FIG. 1: The plot shows the variation of QFI \mathcal{F}_θ^R ; $\theta = \pi/4$ over Rob's cavity in region III as a function of $u := \frac{1}{2}\eta_1/\ln(b/a)$ with $h = 0.01$. The solid curve (black) is for $s = 0$ with $k = \pm 1$. The dashed, dash-dotted and dotted curves are, respectively, for $s = \frac{1}{4}, \frac{1}{2}, \frac{3}{4}$, in $k=1$ (blue) above the solid curve and $k=-1$ (red) below the solid curve.

Rob's cavity in region III. In this case, quantum part of the QFI, $F_\theta = 0$. Thus the only contribution to QFI, \mathcal{F}_θ^R , is

due to its classical part, F_c . Working perturbatively, as a result, we have

$$\begin{aligned} \frac{(p'_1)^2}{p_1} &= \frac{\sin^2 2\theta (1 - f_k h^2)^2}{\cos^2 \theta + (-\cos^2 \theta f_k^{-\nu} + \sin^2 \theta f_k^\nu) h^2} \\ &= 4 \sin^2 \theta [1 + (f_k^{-\nu} - f_k^\nu \tan^2 \theta - 2f_k) h^2], \end{aligned} \quad (40a)$$

$$\frac{(p'_2)^2}{p_2} = 4 \cos^2 \theta [1 + (f_k^\nu - f_k^{-\nu} \cot^2 \theta - 2f_k) h^2], \quad (40b)$$

where f_k^+ , f_k^- given by (28) can be re-written as

$$f_k^+ = \sum_{p \geq 0}^{\infty} |E_1^{k-p}|^2 |A_{pk}^{(1)}|^2, \quad (41a)$$

$$f_k^- = \sum_{q < 0}^{\infty} |E_1^{k-q}|^2 |A_{qk}^{(1)}|^2, \quad (41b)$$

$$f_k : = f_k^+ + f_k^- = \sum_{p=-\infty}^{\infty} |A_{pk}^{(1)}|^2, \quad (41c)$$

with

$$E1 := \exp \left(\frac{i\pi\eta_1}{\ln(b/a)} \right). \quad (42)$$

The QFI \mathcal{F}_θ^R finally reads

$$\mathcal{F}_\theta^R = 4 - 4 \frac{(\sin^2 \theta f_k^\nu + \cos^2 \theta f_k^{-\nu})}{\sin^2 \theta \cos^2 \theta} h^2. \quad (43)$$

Since by definition (41), the relations f_k^ν and $f_k^{-\nu}$ are periodic and non-negative, the QFI over Rob's subsystem is non-negative and is less than or equal to 4 with $0 < \theta < \pi/2$ for the perturbative regime $|k|h \ll 1$. Plots for parameter $\theta = \pi/4$ with $k = \pm 1$ are shown in Fig. 1. Consequently, the QFI of the bipartite composite system, \mathcal{F}_θ , is sub-additive.

V. QUANTUM FISHER INFORMATION FOR THE WERNER STATE

Here, we study the dynamics of QFI for the Werner state from region I to region III. For this purpose, we consider the initial two qubit Werner state in region I given by

$$\begin{aligned} \rho_\nu &= r (\cos \theta |0\rangle\langle 0| + \sin \theta |1_m\rangle\langle 1_m|^\nu) \\ &\quad \times (\cos \theta |0\rangle\langle 0| + \sin \theta |1_m\rangle\langle 1_m|^\nu) + \frac{1-r}{4} \mathbb{I}, \end{aligned} \quad (44)$$

where the parameter r indicates the mixedness of the pure entangled two qubit state and the maximally mixed bipartite state. Considering the perturbed evolution of the state in Rob's cavity from region I to region III in the similar fashion as discussed earlier, we transform the density matrix part confined to Rob's cavity in terms of Rob's Region (III) basis to the order h^2 with the help of (22) and (23). Afterwards we apply the partial trace as given in [19]. The reduced density matrix representation is therefore

$$\begin{aligned} \rho_{W;\nu,k} &= r \left[\cos^2 \theta |0\rangle_A \langle 0| \otimes \left\{ (1 - f_k^{-\nu} h^2) |\tilde{0}\rangle_{III} \langle \tilde{0}| + f_k^{-\nu} h^2 |\tilde{1}_k\rangle_{III} \langle \tilde{1}_k| \right\} + \sin \theta \cos \theta \left\{ (G_k + \mathcal{A}_{kk}^{(2)} h^2) \right. \right. \\ &\quad \times |0\rangle_A^\mu \langle 1_m| \otimes |\tilde{0}\rangle_{III}^\nu \langle \tilde{1}_k| \pm h.c. \left. \right\} + \sin^2 \theta |1_m\rangle_A^{\mu\mu} \langle 1_m| \otimes \left\{ (1 - f_k^\nu h^2) |\tilde{1}_k\rangle_{III}^{\nu\nu} \langle \tilde{1}_k| + f_k^\nu h^2 |\tilde{0}\rangle_{III} \langle \tilde{0}| \right\} \left. \right] \\ &\quad + r^c \left[(|0\rangle_A \langle 0| + |1_m\rangle_A^{\mu\mu} \langle 1_m|) \otimes \left\{ (1 + g_k^\nu h^2) |\tilde{0}\rangle_{III} \langle \tilde{0}| + (1 + g_k^{-\nu} h^2) |\tilde{1}_k\rangle_{III} \langle \tilde{1}_k| \right\} \right], \end{aligned} \quad (45)$$

where r^c and $g_k^{\pm\nu}$ are defined as

$$r^c = \frac{1-r}{4}, \quad (46a)$$

$$g_k^{\pm\nu} = f_k^{\pm\nu} - f_k^{\mp\nu}. \quad (46b)$$

Thus the perturbed density matrix can be expressed as

$$\rho_{W;\nu,k} = \rho_{W;\nu,k}^{(0)} + \rho_{W;\nu,k}^{(2)} h^2. \quad (47)$$

The unperturbed part of the density matrix, $\rho_{W;\nu,k}^{(0)}$, admits the same set of the eigenvectors as obtained for the two mode pure state in Sec. IV corresponding to the respective unperturbed eigenvalues

$$\text{EigenVal}(\rho_{W;\nu,k}^{(0)}) = \{r + r^c, r^c, r^c, r^c\}. \quad (48)$$

Two explicit cases arise for $r = 0$ and $r = 1$. For $r = 0$, the unperturbed density matrix represents the maximally mixed state, with standard basis and degenerate eigenvalue of $p_{1,2,3,4}^{(0)} = 1$. In this case, the QFI with respect to the parameter θ yields a trivial result, $\mathcal{F}_\theta = 0$. For $r = 1$, the situation is exactly the same as described in the Sec. IV. Therefore we restrict the value of r , in the open interval $0 < r < 1$. Using standard perturbation theory, the eigenvalues of the reduced density matrix $\rho_{W;\nu,k}$ in region III are

$$\begin{aligned} \{p_i\} = \Big\{ & r + r^c - r(\cos^2 \theta f_k^{-\nu} + \sin^2 \theta f_k^\nu) h^2 \\ & + r^c (\cos^2 \theta g_k^\nu + \sin^2 \theta g_k^{-\nu}) h^2, \\ & r^c + r^c (\sin^2 \theta g_k^\nu + \cos^2 \theta g_k^{-\nu}) h^2, \\ & r^c + r^c g_k^{-\nu} h^2 + r f_k^{-\nu} \cos^2 \theta h^2, \\ & r^c + r^c g_k^\nu h^2 + r f_k^\nu \sin^2 \theta h^2 \Big\}. \end{aligned} \quad (49)$$

From (49), it can be seen that the perturbed density matrix satisfies the density matrix conditions, $p_i \geq 0; \forall i$ and $\sum_i p_i = 1$. Furthermore, the classical contribution, to the QFI, F_c , vanishes for $0 < r < 1$. This is due to the fact that for each eigenvalue, p_i , the expression $(\partial_\theta p_i)^2/p_i$ has the leading term of order h^4 . Hence, $F_c = 0 + O(h^4)$. However, it is important to note that for the special case of $r = 1$, the classical contribution F_c is non-vanishing and is given by (38). Next, the corresponding perturbed eigenvectors are

$$|\Phi_1\rangle = \frac{1}{\sqrt{N}} \left\{ G_k \left(\cos \theta - \frac{\beta}{r} \sin \theta \right) \ 0 \ 0 \ \sin \theta + \frac{\beta}{r} \cos \theta \right\}, \quad (50a)$$

$$|\Phi_2\rangle = \frac{1}{\sqrt{N}} \left\{ -G_k \left(\sin \theta + \frac{\beta}{r} \cos \theta \right) \ 0 \ 0 \ \cos \theta - \frac{\beta}{r} \sin \theta \right\}, \quad (50b)$$

$$|\Phi_3\rangle = \{ 0 \ 1 \ 0 \ 0 \}, \quad (50c)$$

$$|\Phi_4\rangle = \{ 0 \ 0 \ 1 \ 0 \}, \quad (50d)$$

where $N = 1 + |\beta|^2/r^2$ is the normalization constant and β is defined as

$$\beta = \sin \theta \cos \theta \left[\frac{f_k^{-\nu} - f_k^\nu}{2} + ir \text{Im}(G_k \bar{\mathcal{A}}_{kk}^{(2)}) \right] h^2. \quad (51)$$

The quantum contribution due to individual parts is

$$\begin{aligned} F_{\theta,i} &= 4 \sum_i p_i \mathcal{F}_{\theta,i} \\ &= 4 \left[\frac{1+r}{2} + \left\{ \frac{1+r}{2r} \cos 2\theta (f_k^{-\nu} - f_k^\nu) \right. \right. \\ &\quad \left. \left. - r(\cos^2 \theta f_k^{-\nu} + \sin^2 \theta f_k^\nu) \right\} h^2 \right]. \end{aligned} \quad (52)$$

Next, we have the contribution to the QFI due to the mixture, which is given by

$$\begin{aligned}
F_{\theta;ij} &= 8 \sum_{i \neq j} \mathcal{F}_{\theta;ij} \\
&= 16 \left[\frac{1+2r-3r^2}{8(1+r)} \left\{ 1 + \frac{\cos 2\theta}{r} (f_k^{-\nu} - f_k^{\nu}) h^2 \right\} \right. \\
&\quad + (1-r)r \left\{ \frac{1+3r}{4(1+r)^2} (\sin^2 \theta f_k^{\nu} + \cos^2 \theta f_k^{-\nu}) \right. \\
&\quad \left. \left. - \frac{1}{2(1+r)} (\sin^2 \theta f_k^{-\nu} + \cos^2 \theta f_k^{\nu}) \right\} h^2 \right]. \tag{53}
\end{aligned}$$

The QFI \mathcal{F}_{θ} of the Werner state, (45), with $0 < r < 1$ can therefore be expressed using (52) and (53) as

$$\mathcal{F}_{\theta} = F_{\theta} = F_{\theta;i} - F_{\theta;ij}. \tag{54}$$

It can be noticed that for $r < 1$, the classical contribution, F_c , is zero while the quantum mixture, $F_{\theta;ij}$, is non-zero. However, for $r = 1$, the classical contribution is non-vanishing while the mixture term disappears and we obtain the same result (39) as a special case. Unlike the pure two qubit state case earlier discussed in Sec. IV, it can be seen from (54) that the QFI of the Werner state is affected due to the inertial and non-inertial segments of Rob's cavity motion for mixing parameter $r \in (0, 1)$. Furthermore, the QFI exhibits periodic degradation for $r \in (0, 1)$ as shown in Fig. 2 for the case of $\theta = \pi/4$ and $r = 1/3$. This periodic degradation, however, can be avoided by fine-tuning the duration of inertial and non-inertial trajectories for Rob's cavity.

The effect of mixing parameter r is also investigated and is shown in Fig. 3. As the parameter r increases, the Werner state is tilted towards the pure entangled state and consequently, the QFI of Werner state is less affected by the inertial and non-inertial segments of motion of Rob's cavity.

The QFI distribution over subsystems of Alice's and Rob's cavity modes can be evaluated using a procedure similar to that of Sec. IV C. The QFI \mathcal{F}_{θ}^A over Alice's cavity in region III is given by

$$\mathcal{F}_{\theta}^A = \frac{4r^2 \sin^2 2\theta}{1 - r^2 \cos^2 2\theta}. \tag{55}$$

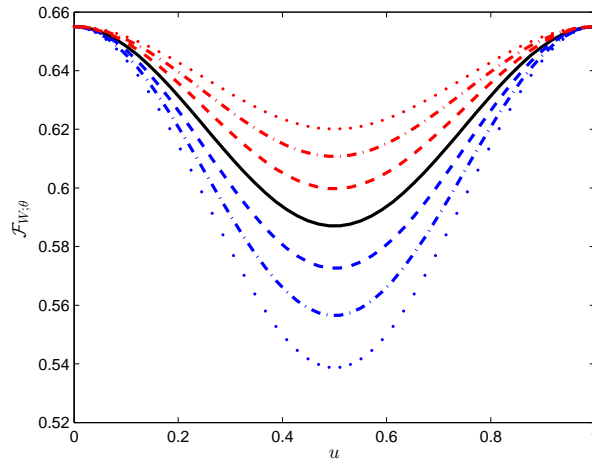


FIG. 2: The plot shows the variation of QFI $\mathcal{F}_{W;\theta}$ at $\theta = \pi/4$ for bipartite Werner state in region III as a function of $u := \frac{1}{2}\eta_1/\ln(b/a)$ with fixed value: $h = 0.01$, $r = 1/3$. The solid curve (black) is for $s = 0$ with $k = \pm 1$. The dashed, dash-dotted and dotted curves are, respectively, for $s = \frac{1}{4}, \frac{1}{2}, \frac{3}{4}$, in $k=1$ (blue) above the solid curve and $k=-1$ (red) below the solid curve.

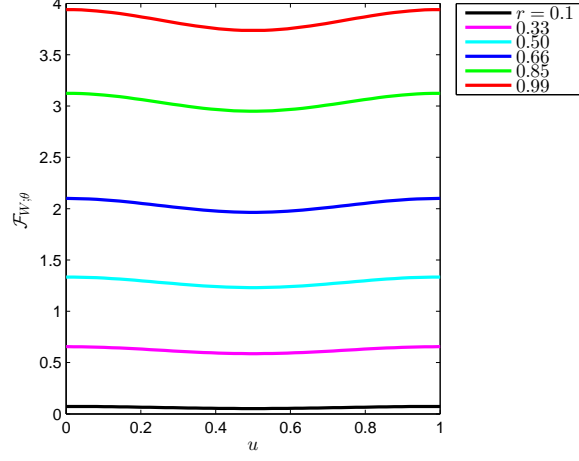


FIG. 3: The plot shows the variation of QFI $\mathcal{F}_{W;\theta}$ at $\theta = \pi/4$ for bipartite Werner state in region III as a function of $u := \frac{1}{2}\eta_1/\ln(b/a)$ with $h = 0.01$. The colored curves from bottom to top correspond to the values of parameter $r = 0.1, 0.33, 0.5, 0.66, 0.85, 0.99$, respectively. The mode in Rob's cavity is kept fixed at $k = 1$ with $s = 0$ for all these curves.

The QFI \mathcal{F}_θ^R over Rob's cavity in region III is given by

$$\mathcal{F}_\theta^R = \frac{4r^2 \sin^2 2\theta}{1 - r^2 \cos^2 2\theta} \left[1 - 2f_k h^2 - \frac{r \cos 2\theta}{1 - r \cos^2 2\theta} \left\{ 4r \times (\cos^2 \theta f_k^{-\nu} - \sin^2 \theta f_k^\nu) + 2(1-r)(f_k^{-\nu} - f_k^\nu) \right\} h^2 \right]. \quad (56)$$

Since the QFI for each cavity mode is non-negative and has a maximum value of 4, therefore the QFI distribution over subsystems is sub-additive.

VI. CONCLUSION AND DISCUSSION

We investigated the effect of relativistic motion on the quantum Fisher information of the $(1+1)$ Dirac field modes confined to cavities. For this purpose, we considered a realistic scheme for two cavities. In our scenario, the modes of a massless Dirac field were confined to the two cavities with Dirichlets boundary conditions, where one of the cavities remained inertial, while the other underwent the segments of inertial and non-inertial motion with uniform acceleration. The acceleration was assumed to be very small and its effects were analyzed in a perturbative regime. We considered θ parameterized two-qubit pure entangled state and a Werner state. In contrast to the degradation of entanglement due to the relativistic motion between the cavities, the quantum Fisher information of the pure composite system \mathcal{F}_θ with respect to parameter θ was found to be invariant under the same conditions. However, in the case of the Werner state, the quantum Fisher information displayed periodic degradation, due to the inertial and non-inertial segments of motion. Furthermore, we found that the quantum Fisher information over Rob's cavity showed periodic degradation behavior depending upon the parameter θ as well as the uniform acceleration for both the two qubit pure state and Werner state. The quantum Fisher information over Alice's cavity remained invariant throughout the motion of Rob's cavity for the two qubit pure state, whereas for the Werner state it was affected by the mixing parameter of the Werner state.

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